Int. J. Solids Structures Vol. 21, No. 1, pp. 97–116, 1985
Printed in the U.S.A.

0020-7683/85 $3.00+.00

ON THE EXISTENCE AND STABILITY CONDITIONS FOR MIXED-HYBRID FINITE ELEMENT SOLUTIONS BASED ON REISSNER'S VARIATIONAL PRINCIPLE†

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(*Received 10 April* 1984)

Abstract—The extensions of Reissner's two-field (stress and displacement) principle to the cases wherein the displacement field is discontinuous and/or the stress field results in unreciprocated tractions, at a finite number of surfaces ("interelement boundaries") in a domain (as, for instance, when the domain is discretized into finite elements), is considered. The conditions for the existence, uniqueness, and stability of mixed-hybrid finite element solutions based on such discontinuous fields, are summarized. The reduction of these global conditions to local ("element") level, and the attendant conditions on the ranks of element matrices, are discussed. Two examples of stable, invariant, least-order elements—a four-node square planar element and an eight-node cubic element—are discussed in detail.

1. INTRODUCTION

On solid mechanics one has, in general, a boundary value problem wherein either the displacements or the tractions, or some components of displacement and the complementary components of tractions, are specified at each point on the boundary. In the geometrically linear theory, to which we restrict ourselves in this paper, single-field variational principles for displacements alone (the so-called principle of virtual work) or for stresses alone (the so-called principle of complementary virtual work) are well known. In obtaining Rayleigh and Ritz-type approximate solutions, using the theorem for displacements, the strain compatibility condition and the displacement boundary condition are satisfied exactly, while the momentum balance conditions and traction boundary conditions are satisfied only approximately. The converse is true for Rayleigh-Ritz solutions using the theorem for stresses. This prompted Reissner [1], "to ask whether it might not be possible to use the calculus of variations for the purpose of obtaining approximate solutions in such a manner that there is no preferential treatment for either one of the two kinds of equations which occur in the theory." Reissner [1] answered the question in the affirmative and stated the now celebrated variational principle governing all admissible states of stress and displacement in the solid domain. In order to be admissible in Reissner's principle [1], the stress (tensor components) and displacement (components) should be differentiable everywhere in the domain and the stress tensor should be symmetric.

In applying Reissner's principle to a finite element model of the solid, with a finite number of interfaces (interelement boundaries), the question naturally arises if one might use a displacement field that is discontinuous at these interfaces and/or use a stress field that results in unreciprocated tractions at the interfaces. The answer to this question is, in the main, the subject of this paper. It is shown that such discontinuous field are permissible, at the expense, however, of introducing additional field variables at the interelement boundaries, thus resulting in several alternative modifications to Reissner's principle. The finite element methods based on these modified principles may be labeled as "mixed-hybrid" methods.

In this paper, abstract statements of the above described modified variational prin-

† This paper is written in honor of Professor Eric Reissner's 70th birthday anniversary. The last-named author expresses his appreciation for the education received from Professor Reissner.

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ciples are given and their equivalence to simultaneous-saddle-point problems is demonstrated. The work of Brezzi [2], who presented the conditions for existence, uniqueness and stability of solutions to a variational problem with a single constraint, is extended in this paper to the present case of simultaneous-saddle-point problems with more than one constraint. The interpretation of these global existence and stability conditions is given, when finite dimensional approximations are used, in terms of the rank of each of the attendant matrices. The reduction of these global conditions to "local" or "element" level conditions is discussed. These "element" level conditions provide unique insights into the development of "mixed-hybrid" finite elements, which are free from "kinematic modes" and other curses that have hitherto plagued these methods. Examples of four-node planar and eight-node three-dimensional hybrid elements are treated to illustrate the applications of the theoretical developments.

2. REISSNER'S PRINCIPLE, AND ITS MODIFICATIONS, IN LINEAR ELASTICITY

We consider a linear elastic solid undergoing infinitesimal deformation. Cartesian coordinates x_i identify material particles in the solid; ϵ_{ij} are the components of the strain tensor; σ_{ij} are components of the stress tensor; u_i are components of the displacement vector; $\bar{f}_i$ are body forces prescribed in the domain Ω of the solid; $\bar{t}_i$ are tractions prescribed at the boundary S_t of the solid; $\bar{u}_i$ are displacements prescribed at S_u of the solid; and $(\ )_{,i}$ denotes a partial derivative with respect to x_i. The field equations are:

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \equiv u_{(i,j)} \text{ in } \Omega \tag{2.1}$$

$$\sigma_{ij,j} + \bar{f}_i = 0; \quad \sigma_{ij} = \sigma_{ji} \text{ in } \Omega \tag{2.2a,b}$$

$$\sigma_{ij} = \partial W/\partial\epsilon_{ij}; \quad \epsilon_{ij} = \partial W_c/\partial\sigma_{ij} \text{ in } \Omega \tag{2.3a,b}$$

$$\sigma_{ij}n_j \equiv t_i = \bar{t}_i \text{ at } S_t \tag{2.4a}$$

and

$$u_i = \bar{u}_i \text{ at } S_u. \tag{2.4b}$$

For the linear elastic materials considered herein, we assume that both W and W_c are positive definite functionals. Henceforth, we shall consider only conservative loading, such that:

$$\bar{u}_i = \frac{\partial U}{\partial t_i} \text{ at } S_u; \quad \bar{t}_i = \frac{\partial T}{\partial u_i} \text{ at } S_t \tag{2.5a}$$

and

$$S_u U S_t = S. \tag{2.5b}$$

Noting that the boundary conditions (2.4a) and (2.4b) involve stresses and displacements, and motivated by the possibility of generating Ritz-type approximate solutions in which both stresses and displacements are simultaneously approximated, Reissner stated† in 1950 [1] the following remarkable two-field (i.e. involving σ_{ij} and u_i) variational principle wherein the strain field is derivable from stress potential W_c as in (2.3b).

Among all differentiable states of stress characterized by the symmetric tensor σ_{ij}, and displacements u_i, the actually occurring state which satisfies eqns (2.1), (2.2a), (2.4a) and (2.4b) is determined by the variational equation:

$$\delta I_{u\sigma} = \delta\left\{\int_\Omega [u_{(i,j)}\sigma_{ij} - W_c(\sigma_{ij}) - \bar{f}_i u_i]\,dv - \int_{S_t} \bar{t}_i u_i\,ds - \int_{S_u} (u_i - \bar{u}_i)t_i\,ds\right\} = 0. \tag{2.6}$$

† For a historical account of this development, see Reissner [3].

We now consider the case when a solid is discretized, for purposes of generating an approximate solution, into a number of finite elements. Following Reissner, we explore the possibility of approximating both stresses and displacements simultaneously. Let $\Omega = \sum_m \Omega_m$, where Ω_m is the mth element, with a boundary $\partial\Omega_m$. In general,

$$\partial\Omega_m = \rho_m + S_{tm} + S_{um} \tag{2.7}$$

wherein, ρ_m is the inter-element boundary, and S_{tm} and S_{um} are those segments of $\partial\Omega_m$ which are in common with the external boundary segments S_t and S_u, respectively. The field equations for the elasticity problem may now be stated for the finite element assembly. Evidently, in each Ω_m, eqns (2.1)–(2.3) must be satisfied. Likewise, eqns (2.4a) and (2.4b) should be obeyed at S_{tm} and S_{um}, respectively. In addition, at the inter-element boundaries, the following conditions must be satisfied:

$$(\sigma_{ij}n_j)^+ + (\sigma_{ij}n_j)^- = 0 \text{ at } \rho_m \tag{2.8}$$

$$u_i^+ = u_i^- \text{ at } \rho_m. \tag{2.9}$$

In eqns (2.8) and (2.9) the superscripts (+) and (−) denote, arbitrarily, the two "sides" of ρ_m. Equations (2.8) and (2.9) are, respectively, the conditions of "traction reciprocity" and "displacement compatibility" at the inter-element interfaces.

We now assume that (2.3b) is satisifed *a priori*, as a physical relation between strains and stresses, i.e. $\epsilon_{ij} \equiv \partial W_c/\partial\sigma_{ij}$. Henceforth, we restrict our attention to a differentiable field u_i and a differentiable symmetric field σ_{ij} within each Ω_m; however, in general, u_i and σ_{ij} do not obey eqns (2.8) and (2.9), unless so specified.

Consider an isolated segment AB (in a two-dimensional example) of the inter-element boundary between elements Ω_m and Ω_{m+1}. Note that AB is a part of both ρ_m (with unit outward normal n_i^-) and ρ_{m+1} (with unit outward normal $n_i^+ (= -n_i^-)$). The weak form of (2.9) may be stated:

$$\int_{AB} (u_i^+ - u_i^-)\tau_i \, dS = 0 \tag{2.10}$$

where τ_i are test functions at AB. If we redefine τ_i for each side of AB such that:

$$\tau_i^+ = \bar{\tau}_{ip}^+ ; \quad \tau_i^- = \bar{\tau}_{ip}^- ; \quad \bar{\tau}_{ip}^+ + \bar{\tau}_{ip}^- = 0 \text{ at } \rho_m \tag{2.11}$$

where $\bar{\tau}_{ip}^{\pm}$ is physically a reciprocated traction field at the inter-element boundary, the constraint eqn (2.10) at *all* inter-element boundaries may be written as:

$$\sum_m \int_{\rho m} u_i\bar{\tau}_{ip} \, ds = 0. \tag{2.12}$$

On the other hand, one may introduce a unique displacement field $\tilde{u}_{ip}$ at the inter-element boundary as an additional variable and enforce eqn (2.9) by setting:

$$u_i^+ = \tilde{u}_{ip}^+ \text{ and } u_i^- = \tilde{u}_{ip}^- \text{ at } AB \text{ (i.e. } \tilde{u}_{ip}^+ = \tilde{u}_{ip} = \tilde{u}_{ip}^- \text{ at } AB) \tag{2.13}$$

In this case, eqn (2.10) may be replaced by:

$$\int_{AB} \{(u_{ip}^+ - \tilde{u}_{ip})\tau_{ip}^+ + (u_i^- - \tilde{u}_{ip})\tau_{ip}^-\} \, ds = 0 \tag{2.14}$$

where $\tau_{ip}^+ + \tau_{ip}^- \neq 0$. Now eqn (2.12) may be replaced by:

$$\sum_m \int_{\rho_m} (u_i - \tilde{u}_{ip})\tau_{ip} \, ds = 0. \tag{2.15}$$

Turning now to eqn (2.8), if a unique test function $\tilde{v}_{ip}$ is introduced for each inter-element boundary segment AB, we may write the weak form of eqn (2.8) as:

$$\int_{AB} [(n_j\sigma_{ij})^+ + (n_j\sigma_{ij})^-]\tilde{v}_{ip}\,ds = 0 \quad (2.16a)$$

or

$$\sum_m \int_{\rho_m} (n_j\sigma_{ij}\tilde{v}_{ip})\,ds = 0 \quad (2.16b)$$

where $\tilde{v}_{ip}^+ = \tilde{v}_{ip} = \tilde{v}_{ip}^-$. On the other hand, eqn (2.8) may also be enforced by setting:

$$(\sigma_{ij}n_j)^+ - \bar{t}_{ip}^+ = 0; \quad (\sigma_{ij}n_j)^- - \bar{t}_{ip}^- = 0 \text{ at } AB \quad (2.17a)$$

where

$$\bar{t}_{ip}^+ + \bar{t}_{ip}^- = 0. \quad (2.17b)$$

Thus, eqn (2.16a) and (2.16b) may be replaced by:

$$\int_{AB} \{(\sigma_{ij}n_j - \bar{t}_{ip})^+ v_{ip}^+ + (\sigma_{ij}n_j - \bar{t}_{ip})^- v_{ip}^-\}\,ds = 0 \quad (2.18a)$$

or

$$\sum_m \int_{\rho_m} (\sigma_{ij}n_j - \bar{t}_{ip})v_{ip}\,ds = 0 \quad (2.18b)$$

where $v_{ip}^+ \neq v_{ip}^-$. An examination of eqns (2.12, 2.15, 2.16) and (2.18) reveals that the following four possibilities exist for writing a combined weak form of eqns (2.8) and (2.9).

Case 1. Suppose eqn (2.9) is satisfied *a priori*. Thus, we may take $\tilde{v}_{ip} \equiv v_i$, where the test function v_i is identical in form to the trial function u_i that obeys eqn (2.9) *a priori*. Thus, eqn (2.16) is replaced by

$$\sum_m \int_{\rho_m} \sigma_{ij}n_jv_i\,ds = 0 \quad (2.19)$$

where, clearly the constraint, $v_i^+ = v_i^-$ at ρ_m, is satisfied *a priori*.

Case 2. Suppose eqn (2.9) is *not* satisfied *a priori*, and that eqn (2.12) is used to enforce eqn (2.9) in the weak sense. Noting the similarity of the functions $\bar{\tau}_{ip}$ as in eqn (2.11) and $\bar{t}_{ip}$ in eqn (2.17b), we may use eqn (2.18b) to enforce eqn (2.8) in the weak sense. We also note the similarity of the functions v_{ip}^+ and v_{ip}^- to the trial functions u_i assumed in each element such that (2.9) is not satisfied *a priori*. Thus, we write the combined weak form of eqns (2.8) and (2.9) as:

$$\sum_m \left\{\int_{\rho_m} (n_j\sigma_{ij} - \bar{t}_{ip})v_i\,ds - \int_{\rho_m} u_i\bar{\tau}_{ip}\,ds\right\} \quad (2.20)$$

where $v_i^+ \neq v_i^-$ and $u_i^+ \neq u_i^-$ at ρ_m.

Case 3. Here again, eqn (2.9) is supposed to be satisfied only *a posteriori*, through eqn (2.15); and eqn (2.8) is enforced *a posteriori* through eqn (2.16). Noting that τ_{ip}^+ and τ_{ip}^- are of the same form as the trial functions $(\sigma_{ij}n_j)^+$ and $(\sigma_{ij}n_j)^-$, respectively, which do not obey eqn (2.8), and that $\tilde{u}_{ip}$ and $\tilde{v}_{ip}$ are of the same form, we write, for the combined weak forms of eqns (2.8) and (2.9):

$$+\sum_m \left\{ \int_{\rho_m} n_j\sigma_{ij}\bar{v}_{i\rho}\,\mathrm{d}s - \int_{\rho_m} (u_i - \tilde{u}_{i\rho})\tau_i\,\mathrm{d}s \right\} = 0 \tag{2.21}$$

where τ_i belongs to the same function space as $\sigma_{ij}n_j \equiv t_i$.

Case 4. Here, eqn (2.9) is enforced *a posteriori* through eqn (2.15). We introduce another independent set of functions $t_{i\rho}$, of the same type as $\tau_{i\rho}$, i.e. $t_{i\rho}^+ + t_{i\rho}^- \neq 0$ at AB. The traction reciprocity condition eqn (2.8) may be stated in weak form, as:

$$\sum_m \left\{ \int_{\rho_m} (n_j\sigma_{ij} - t_{i\rho})\cdot v_{i\rho}\,\mathrm{d}s + \int_{\rho_m} t_{i\rho}\cdot \bar{v}_{i\rho}\,\mathrm{d}s \right\} = 0 \tag{2.22}$$

where $v_{i\rho}^+ \neq v_{i\rho}^-$, but $\bar{v}_{i\rho}^+ = \bar{v}_{i\rho}^- = \bar{v}_{i\rho}$. Now the combined weak form of eqns (2.8) and (2.9) may be stated as:

$$\sum_m \left\{ \int_{\rho_m} (n_j\sigma_{ij} - t_{i\rho})v_i\,\mathrm{d}s + \int_{\rho_m} t_{i\rho}\bar{v}_{i\rho}\,\mathrm{d}s - \int_{\rho_m} (u_i - \tilde{u}_{i\rho})\cdot\tau_{i\rho}\,\mathrm{d}s \right\} = 0. \tag{2.23}$$

Now $v_i^+ \neq v_i^-$, and hence v_i is of the same category as u_i at ρ_m.

The essential ideas for the above four cases of modifying Reissner's variational principle to account for trial functions σ_{ij} and u_i that *do not* obey the inter-element compatibility and traction reciprocity, i.e. eqns (2.8) and (2.9), were first presented to by Atluri [4] (see also [5]).

We now consider the weak forms of the finite element counterparts of eqns (2.1)–(2.5). The weak form of eqn (2.2) (when σ_{ij} is taken to be symmetric *a priori*), is:

$$\sum_m \int_{\Omega_m} (\sigma_{ij,j} + \bar{f}_i)\cdot\mu_i\,\mathrm{d}v = \sum_m \int_{\partial\Omega_m} n_j\sigma_{ij}\mu_i\,\mathrm{d}s + \int_{\Omega_m} [\bar{f}_i\mu_i - \sigma_{ij}\mu_{(i,j)}]\,\mathrm{d}v\,. \tag{2.24}$$

The combined weak forms of eqns (2.1)–(2.5) may now be written as:

$$0 = \sum_m \left\{ \int_{\Omega_m} \left[\left(\frac{\partial W_c}{\partial\sigma_{ij}} - u_{(i,j)} \right)\lambda_{ij} + \bar{f}_i\mu_i - \sigma_{ij}\mu_{(i,j)} \right]\mathrm{d}v + \int_{\partial\Omega_m} n_j\sigma_{ij}\mu_i\,\mathrm{d}s \right.$$
$$\left. + \int_{S_{tm}} (n_j\sigma_{ij} - \bar{t}_i)\nu_i\,\mathrm{d}s + \int_{S_{um}} (u_i - \bar{u}_i)\theta_i\,\mathrm{d}s \right\}. \tag{2.25}$$

The combined weak form of eqns (2.1)–(2.3) and eqns (2.7) and (2.8) may now be written for each of the four cases listed earlier:

Case 1. Here one may choose the test functions such that: $\mu_i = -v_i$; $\nu_i = v_i$; $\theta_i = -\tau_i = -\tau_{ij}n_j$; $\lambda_{ij} = -\tau_{ij}$. Adding eqns (2.19) and (2.25), we obtain:

$$0 = \sum_m \left\{ \int_{\Omega_m} \left[\left(u_{(i,j)} - \frac{\partial W_c}{\partial\sigma_{ij}} \right)\tau_{ij} + \sigma_{ij}v_{(i,j)} - \bar{f}_i v_i \right]\mathrm{d}v - \int_{S_{tm}} \bar{t}_i v_i\,\mathrm{d}s \right.$$
$$\left. - \int_{S_{um}} (u_i - \bar{u}_i)\tau_{ij}n_j\,\mathrm{d}s - \int_{S_{um}} \sigma_{ij}n_j v_i\,\mathrm{d}s \right\}. \tag{2.26}$$

Now, using the prevalant notation of calculus of variations, we may denote: $\tau_{ij} = \delta\sigma_{ij}$ and $v_i = \delta u_i$. Thus, eqn (2.26) may be written:

$$0 = \delta\sum_m \left\{ \int_{\Omega_m} [-W_c(\sigma_{ij}) + \sigma_{ij}u_{(i,j)} - \bar{f}_i u_i]\,\mathrm{d}v - \int_{S_t} \bar{t}_i u_i\,\mathrm{d}s - \int_{S_u} (u_i - \bar{u}_i)\sigma_{ij}n_j\,\mathrm{d}s \right\}.$$

(2.27)

Thus, we obtain the interesting result that for all trial functions u_i that obey eqn (2.9) and all symmetric trial functions σ_{ij} that *do not* obey eqn (2.8), the variational functional governing the finite element method is exactly the same as that originally stated by Reissner [1] except that the relevant integrals in eqn (2.6) are replaced by the sums of their counterparts evaluated over each element.

Case 2. Choose μ_i, ν_i, θ_i, and λ_{ij} as in Case 1; and set $\tilde{u}_{i\rho} = \bar{u}_i$ at S_{um}. Then, adding eqns (2.20) and (2.25), we obtain:

$$0 = \sum_m \left\{ \int_{\Omega_m} (\ )\mathrm{d}v - \int_{S_{tm}} \bar{t}_i v_i \,\mathrm{d}s - \int_{S_{um}} (u_i - \tilde{u}_{i\rho})\tau_{ij} n_j \,\mathrm{d}s \right.$$

$$\left. - \int_{S_{um}} v_i \sigma_{ij} n_j \,\mathrm{d}s - \int_{\rho_m} \tilde{t}_{i\rho} v_i \,\mathrm{d}s - \int_{\rho_m} \tilde{\tau}_{i\rho} u_i \,\mathrm{d}s \right\} \tag{2.28}$$

$$= \delta \sum_m \left\{ \int_{\Omega_m} (\ )\mathrm{d}v - \int_{S_t} \bar{t}_i u_i \,\mathrm{d}s - \int_{S_u} (u_i - \bar{u}_i)\sigma_{ij} n_j \,\mathrm{d}s \right.$$

$$\left. - \int_{\rho_m} u_i \tilde{t}_{i\rho} \,\mathrm{d}s \right\}. \tag{2.29}$$

The integrands in the volume V_m in eqns (2.28) and (2.29) are identical to those in eqns (2.26) and (2.27), respectively.

Case 3. Choosing μ_i, ν_i, θ_i, and λ_{ij} as in Case 1, and adding eqn (2.21) to eqn (2.25), we obtain:

$$0 = \sum_m \left\{ \int_{\Omega_m} (\ )\mathrm{d}v - \int_{S_{tm}} \bar{t}_i v_i \,\mathrm{d}s - \int_{S_{um}} (u_i - \bar{u}_i) n_j \tau_{ij} \,\mathrm{d}s \right.$$

$$- \int_{S_{um}} v_i n_j \sigma_{ij} \,\mathrm{d}s - \int_{\rho_m} (u_i - \tilde{u}_{i\rho}) n_j \tau_{ij} \,\mathrm{d}s$$

$$\left. - \int_{\rho_m} (v_i - \tilde{v}_{i\rho}) n_j \sigma_{ij} \,\mathrm{d}s \right\} \tag{2.30}$$

$$= \delta \sum_m \left\{ \int_{\Omega_m} (\ )\mathrm{d}v - \int_{S_{tm}} \bar{t}_i u_i \,\mathrm{d}s - \int_{S_{um}} (u_i - \bar{u}_i) n_j \sigma_{ij} \,\mathrm{d}s \right.$$

$$\left. - \int_{\rho_m} (u_i - \tilde{u}_{i\rho}) n_j \sigma_{ij} \,\mathrm{d}s \right\} \tag{2.31}$$

wherein, the volume integrals are the same as before.

Case 4. Choosing μ_i, ν_i, and λ_{ij} as in Case 1 and $\theta_i = -\tau_{i\rho}$, and adding (2.22) to (2.25), we obtain:

$$0 = \sum_m \left\{ \int_{\Omega_m} (\ )\mathrm{d}v - \int_{S_{tm}} \bar{t}_i v_i \,\mathrm{d}s - \int_{S_{um}} (u_i - \bar{u}_i)\tau_{i\rho} \,\mathrm{d}s \right.$$

$$\left. - \int_{S_{um}} v_i t_{i\rho} \,\mathrm{d}s - \int_{\rho_m} (v_i - \tilde{v}_{i\rho}) t_{i\rho} \,\mathrm{d}s - \int_{\rho_m} (u_i - \tilde{u}_{i\rho})\tau_{i\rho} \,\mathrm{d}s \right\} \tag{2.32}$$

$$= \delta \sum_m \left\{ \int_{\Omega_m} (\ )\mathrm{d}v - \int_{S_{tm}} t_i u_i \,\mathrm{d}s - \int_{S_{um}} (u_i - \bar{u}_i) t_{i\rho} \,\mathrm{d}s \right.$$

$$\left. - \int_{\rho_m} (u_i - \tilde{u}_{i\rho}) t_{i\rho} \,\mathrm{d}s \right\} \tag{2.33}$$

wherein, once again, the volume integrals are the same as before.

We now consider the stability conditions for finite element methods based on eqns (2.26), (2.28), (2.30) and (2.32), respectively. To avoid repetition, however, we consider only representative examples of mixed-hybrid finite element methods based on eqns (2.30) and (2.32).

3. AN ABSTRACT STATEMENT OF MODIFIED REISSNER'S PRINCIPLE

We define the following bilinear forms and linear functionals:

$$a(\sigma, \tau) = \sum_m \int_{\Omega_m} \frac{\partial W_c}{\partial \sigma_{ij}} \tau_{ij}\, dv; \qquad b(\tau, v) = \sum_m \int_{\Omega_m} \tau_{ij} v_{(i,j)}\, dv \tag{3.1,2}$$

$$c(v, t_\rho) = \sum_m \int_{\rho_m} v_i t_{i\rho}\, ds; \qquad d(t_\rho, \tilde{v}_\rho) = \sum_m \int_{\rho_m} t_{i\rho} \tilde{v}_{i\rho}\, d\rho \tag{3.3,4}$$

$$\langle f, v\rangle = \sum_m \left\{ \int_{\Omega_m} \bar{f}_i v_i\, dv + \int_{S_{tm}} \bar{t}_i v_i\, ds \right\}. \tag{3.5}$$

Assuming that the trial function u_i satisfies the prescribed condition at S_u, and thus the test function $v_i = 0$ at S_u, it is easily seen that weak form eqn (2.32) has the following abstract statement:

Find $(\sigma, u, t_\rho, \tilde{u}_\rho) \in T \times V_0 \times T(\rho) \times \tilde{V}(\rho)$ such that:

$$\begin{aligned} a(\sigma, \tau) - b(\tau, u) &= 0 && \forall \tau \in T \\ b(\sigma, v) - c(v, t_\rho) &= \langle f, v\rangle && \forall v \in V_0 \\ c(u, \tau_\rho) - d(\tau_\rho, \tilde{u}_\rho) &= 0 && \forall \tau_\rho \in T(\rho) \\ d(t_\rho, \tilde{v}_\rho) &= 0 && \forall v_\rho \in \tilde{V}(\rho) \end{aligned} \tag{3.6}$$

where

$$V_0 = \{v \in H^1(\Omega_m); \quad v_i = 0 \text{ on } S_{um}, \forall m\}. \tag{3.7}$$

THEOREM 3.1

For the type of materials, namely, those for which W_c is a positive definite functional, it is seen that $a(\cdot, \cdot)$ is symmetric and positive definite. Then, problem (3.6) is equivalent to the simultaneous saddle-point problem:

Find $(\sigma, u, t_\rho, \tilde{u}_\rho) \in T \times V_0 \times T(\rho) \times \tilde{V}(\rho)$ such that:

$$\begin{aligned} \mathcal{L}(\sigma, v, t_\rho, \tilde{u}_\rho) \le \mathcal{L}(\sigma, u, t_\rho, \tilde{u}_\rho) &\le \mathcal{L}(\tau, u, t_\rho, \tilde{u}_\rho) && \forall v, \tau \\ \mathcal{L}(\sigma, u, \tau_\rho, \tilde{u}_\rho) \le \mathcal{L}(\sigma, u, t_\rho, \tilde{u}_\rho) &\le \mathcal{L}(\tau, u, t_\rho, \tilde{u}_\rho) && \forall \tau_\rho, \tau \\ \mathcal{L}(\sigma, u, t_\rho, \tilde{v}_\rho) \le \mathcal{L}(\sigma, u, t_\rho, \tilde{u}_\rho) &\le \mathcal{L}(\tau, u, t_\rho, \tilde{u}_\rho) && \forall \tilde{v}_\rho, \tau \end{aligned} \tag{3.8}$$

where $\mathcal{L}(\sigma, u, t_\rho, \tilde{u}_\rho)$ is given by:

$$\mathcal{L} = \tfrac{1}{2}a(\sigma, \sigma) - b(\sigma, u) + \langle f, u\rangle + c(u, t_\rho) - d(t_\rho, \tilde{u}_\rho) \tag{3.9}$$

as shown in eqn (2.33).

Proof

Suppose that (σ, u, t_ρ, and $\tilde{u}_\rho$) is the solution of problem (3.6). We then have:

$$\begin{aligned} &\mathcal{L}(\sigma + \tau, u, t_\rho, \tilde{u}_\rho) - \mathcal{L}(\sigma, u, t_\rho, \tilde{u}_\rho) \\ &= \tfrac{1}{2}a(\tau, \tau) + [a(\sigma, \tau) - b(\tau, u)] \\ &= \tfrac{1}{2}a(\tau, \tau) \ge 0. \end{aligned}$$

Thus,

$$\mathscr{L}(\sigma, u, t_\rho, \bar{u}_\rho) \leq \mathscr{L}(\tau, u, t_\rho, \bar{u}_\rho) \qquad \forall\tau \in T. \tag{3.10}$$

Moreover,

$$\mathscr{L}(\sigma, u + v, t_\rho, \bar{u}_\rho) - \mathscr{L}(\sigma, u, t_\rho, \bar{u}_\rho) = -b(\sigma, v) + \langle f, v\rangle + c(v, t_\rho) = 0.$$

Thus,

$$\mathscr{L}(\sigma, v, t_\rho, \bar{u}_\rho) \leq \mathscr{L}(\sigma, u, t_\rho, \bar{u}_\rho) \qquad \forall v \in V_0. \tag{3.11}$$

Equations (3.10) and (3.11) imply the first inequality of eqn (3.8). Similarly, we can prove the other inequalities.

Now, to prove the converse, suppose that $(\sigma, u, t_\rho, \bar{u}_\rho)$ is the solution of eqn (3.8). Then, the inequality

$$\mathscr{L}(\sigma, u, t_\rho, \bar{v}_\rho) \leq \mathscr{L}(\sigma, u, t_\rho, \bar{u}_\rho) \qquad \forall\bar{v}_\rho \in \tilde{V}(\rho)$$

implies that:

$$\mathscr{L}(\sigma, u, t_\rho, \bar{u}_\rho + \bar{v}_\rho) - \mathscr{L}(\sigma, u, t_\rho, \bar{u}_\rho) = -\langle Dt_\rho, \bar{v}_\rho\rangle \leq 0 \qquad \forall\bar{v}_\rho \in \tilde{V}(\rho).$$

In view of the linearity of Dt_ρ, the above inequality leads to:

$$d(t_\rho, \bar{v}_\rho) = 0 \qquad \forall\bar{v}_\rho \in \tilde{V}(\rho).$$

The second and third equations of problem (3.6) may similarly be proved. To prove the first equation of (3.6), consider the inequality:

$$\mathscr{L}(\sigma + \tau, u, t_\rho, \bar{u}_\rho) - \mathscr{L}(\sigma, u, t_\rho, \bar{u}_\rho) = \tfrac{1}{2}a(\tau, \tau) + [a(\sigma, \tau) - b(\tau, u)] \geq 0.$$

Letting $\tau_{ij} = \epsilon s_{ij}$, with ϵ being an arbitrarily small number and $s \in T$, we obtain:

$$a(\sigma, s) - b(s, u) \geq -\frac{\epsilon}{2}\, a(s, s) \qquad \forall\, \epsilon, s,$$

from which the first equation of (3.8) follows.

4. STABILITY CONDITIONS FOR GENERAL MIXED-HYBRID FINITE ELEMENTS BASED ON MODIFIED REISSNER'S PRINCIPLE:

Let $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ be continuous bilinear forms. Brezzi [2] considered an arbitrary variational problem, wherein a single constraint condition is enforced through a Lagrange multiplier, of the form:

Find $(u, q) \in V \times P$, such that:

$$\begin{aligned} a(u, v) - b(v, q) &= \langle f, v\rangle \qquad \forall v \in V \\ b(u, p) &= \langle g, p\rangle \qquad \forall p \in P. \end{aligned} \tag{4.1}$$

Brezzi [2] proved the following theorem (see also [6]):

THEOREM 4.1

Problem (4.1) has a unique solution if the following conditions are satisisfied:

$$\sup_{\forall v \in V} \frac{b(v, p)}{\| v \|_V} \geq \beta \| p \|_P \qquad \forall p \in P \tag{4.2a}$$

$$a(v, v) \geq \alpha \| v \|^2 \qquad \forall v \in \mathrm{Ker}(B) \tag{4.2b}$$

where α and β are positive numbers, and the subspace Ker(B) is defined by:

$$\text{Ker}(B) = \{v \in V,\ b(v, p) = 0 \qquad \forall p \in P\}. \tag{4.2c}$$

Furthermore, one has the following estimate:

$$\| u \|_V + \| q \|_P \leq c(\| f \|_{V^*} + \| g \|_{P^*}) \tag{4.2d}$$

Remark

The condition (4.2a) is sometimes referred to as the Ladyzhenskaya [7], Babuska [8], and Brezzi [1] condition and is equivalent to the following inf-sup condition:

$$\inf_{\forall p} \sup_{\forall v} \frac{b(v, p)}{\| v \|_V \| p \|_P} > \beta \geq 0. \tag{4.3a}$$

We also state a fundamental lemma due to Girault and Raviart [9] as follows:
Lemma 4.1. Condition (4.3a) is equivalent to the properties:

(i) the operator B^* is an isomorphism from P onto $[\text{Ker}(B)]^0$. Therefore,

$$\| B^*P \|_{V^*} \geq \beta \| p \|_P \qquad \forall p \in P. \tag{4.3b}$$

(ii) the operator B is an isomorphism from $[\text{Ker}(B)]_\perp$ onto P^*. Therefore,

$$\| Bv \|_{P^*} \geq \beta \| v \|_V \qquad \forall v \in [\text{Ker}(B)]_\perp \tag{4.3c}$$

where,

$$[\text{Ker}(B)]_\perp = \{v \in V,\ (v, v_0) = 0 \qquad \forall v_0 \in \text{Ker}(B)\} \tag{4.4a}$$

and

$$[\text{Ker}(B)]^0 = \{g \in V^*,\ \langle g, v_0 \rangle = 0 \qquad \forall v_0 \in \text{Ker}(B)\} \tag{4.4b}$$

where (u, v) indicates an inner product of u and v, and $\langle g, v \rangle$ the linear functional with the linear operator $\langle g, \cdot \rangle$.

Remark

The estimate (4.2d) endows the property to the solution (u, q) that it is stable with respect to perturbations in initial data f and g.

The work of Brezzi [2] has recently been applied by Ying and Atluri [10] to the problem of a hybrid-finite element solution of incompressible viscous at zero Re. number (Stokes' flow), which involves a pair of Lagrange multipliers. Hence, Ying and Atluri [10] state two conditions of the type of the present (4.2a) for the uniqueness of the problem considered in [10]. We now state the stability conditions for the present problem, (3.6):

THEOREM 4.2

Problem (3.6) has a unique solution if the following conditions are met:

$$\sup_{\forall t_\rho \in T(\rho)} \frac{d(t_\rho, v_\rho)}{| t_\rho |_{T(\rho)}} \geq \beta \, | \tilde{v}_\rho |_{\tilde{V}(\rho)} \qquad \forall \tilde{v}_\rho \in \tilde{V}(\rho) \tag{4.5a}$$

$$\sup_{\forall v \in V_0} \frac{c(v, t_\rho)}{| v |_{V_0}} \geq \beta \, | t_\rho |_{T(\rho)} \qquad \forall t_\rho \in \text{Ker}(D) \tag{4.5b}$$

$$\sup_{\forall \tau \in T} \frac{b(\tau, v)}{| \tau |_T} \geq \beta \, | v |_{V_0} \qquad \forall v \in \text{Ker}(C) \tag{4.5c}$$

and

$$a(\tau, \tau) \geq \alpha \mid \tau \mid_T^2 \qquad \forall \tau \in \mathrm{Ker}(B) \tag{4.5d}$$

where $d(t_\rho, \bar{v}_\rho)$, $c(v, t_\rho)$, $b(\tau, v)$, and $a(\tau, \tau)$ are defined in eqns (3.4), (3.3), (3.2), and (3.1) respectively, and the various kernels are defined as follows:

$$\mathrm{Ker}(D) = \{\tau_\rho \in T(\rho); \quad d(\tau_\rho, \bar{v}_\rho) = 0 \quad \forall \bar{v}_\rho \in V(\rho)\} \tag{4.5e}$$

$$\mathrm{Ker}(C) = \{v \in V; \quad c(v, t_\rho) = 0 \quad \forall t_\rho \in \mathrm{Ker}(D)\} \tag{4.5f}$$

$$\mathrm{Ker}(B) = \{\tau \in T; \quad b(\sigma, v) = 0 \quad \forall v \in \mathrm{Ker}(C)\}. \tag{4.5g}$$

Moreoever, when β is positive, one has the estimate: .

$$\mid \sigma \mid_T + \mid u \mid_V + \mid t_\rho \mid_{T(\rho)} + \mid \bar{u}_\rho \mid_{V(\rho)} \leq C \mid f \mid_{T^*} \tag{5.5h}$$

Proof

According to condition eqn (4.5a) and Lemma 4.1 (i), the operator D^* is an isomorphism from $\bar{V}(\rho)$ onto $[\mathrm{Ker}(D)]^0$. Thus, the third equation in (3.6) has a unique solution $\bar{u}_\rho \in \bar{V}(\rho)$, if and only if $c(u, \cdot)$ belongs to $[\mathrm{Ker}(D)]^0$, i.e.

$$c(u, \tau_\rho) = 0 \qquad \forall \tau_\rho \in \mathrm{Ker}(D).$$

From the fourth equation in (3.6) it is seen that $t_\rho \in \mathrm{Ker}(D)$. Therefore, the problem (3.6) is reduced to the following abstract problem:

Find $(\sigma, u, t_\rho) \in T \times V_0 \times \mathrm{Ker}(D)$ such that:

$$\begin{aligned} a(\sigma, \tau) - b(\tau, u) &= 0 \qquad \forall \tau \in T \\ b(\sigma, v) - c(v, t_\rho) &= \langle f, v \rangle \qquad \forall v \in V_0 \\ c(u, \tau_\rho) &= 0 \qquad \forall \tau_\rho \in \mathrm{Ker}(D). \end{aligned} \tag{4.6}$$

Similarly, noting condition (4.5b), one may use reasoning identical to above, and may reduce (4.6) further to the abstract problem:

Find $(\sigma, u) \in T \times \mathrm{Ker}(C)$, such that:

$$\begin{aligned} a(\sigma, \tau) - b(\tau, u) &= 0 \qquad \forall \tau \in T \\ b(\sigma, v) &= \langle f, v \rangle \qquad \forall v \in \mathrm{Ker}(C) \end{aligned} \tag{4.7}$$

where,

$$\mathrm{Ker}(C) = \{v \in V_0, \quad c(v\ \tau_\rho) = 0, \quad \forall \tau_\rho \in \mathrm{Ker}(D)\}.$$

According to Theorem (4.1), (σ, u) represent a unique solution to (4.7) if, and only if, (4.5c) and (4.5d) are met. Likewise, $(\sigma, u, t_\rho, \bar{u}_\rho)$ represent a unique solution to (3.6) provided the conditions (4.5a–d) are met. From Theorem (4.1), as applied to (4.7), we have, if (4.5c and d) are met,

$$\| \sigma \|_T + \| u \|_{V_0} \leq C_1 \| f \|_{V^*}. \tag{4.8}$$

When (4.5a and b) are met, we have, from Lemma 4.1(i),

$$\| \bar{u}_\rho \|_{V(\rho)} \leq \beta^{-1} \| D^* \bar{u}_\rho \|_{T(\rho)^*} = \beta^{-1} \| CU \|_{T^*(\rho)} \leq \beta^{-1} \| C \| \| u \|_V$$

and

$$\| t_p \|_{T(p)} \leq \beta^{-1} \| C^* t_p \|_{V^*} = \beta^{-1} \| B\sigma - f \|_{V^*} \leq \beta^{-1} (\| B \| \| \sigma \|_T + \| f \|_{V^*}).$$

Hence,

$$\| \tilde{u}_p \|_{\tilde{V}(p)} + \| t_p \|_{T(p)} \leq \beta^{-1} \max\{\| C \|, \| B \|\}(\| \sigma \|_T + \| u \|_{V_0}) + \beta^{-1} \| f \|_{V^*} \quad (4.9)$$

The estimate (4.5h) follows from (4.8) and (4.9).

5. THE DISCRETE STABILITY CONDITION AND THE RANK CONDITION

The key conditions for the existence, uniqueness, convergence, and stability of the hybrid-mixed finite element approximations based on Reissner's principle, when finite dimensional (discrete) approximations are introduced, are of the type:

$$\inf_{\forall p \in P_h} \sup_{\forall v \in V_h} \frac{b(v, p)}{\| v \|_V \| p \|_P} \geq \beta > 0 \quad (5.1)$$

where V_h and P_h are finite dimensional subspaces of Hilbert Spaces V and P, respectively. The dimensions of V_h and P_h are assumed to be:

$$\dim V_h = m; \qquad \dim P_h = n. \quad (5.2)$$

Then the operator B defined by:

$$\langle Bv, p \rangle \triangleq b(v, p) \qquad \forall (v, p) \in V_h \times P_H \quad (5.3)$$

has an $(n \times m)$ matrix representation. We shall denote this matrix also by B. Thus,

$$b(v, p) = \underset{(1 \times n)}{p^t} \; \underset{(n \times m)}{B} \; \underset{(m \times 1)}{v} .$$

From Lemma (4.1), the operator B is an isomorphism from $[\mathrm{Ker}(B)]_\perp$ on to space P_h^*. Since P_h^* is now the dual space of a finite dimensional subspace of Hilbert space P, we have $P_h^* = P_h$. Then, B is also regarded as an isomorphism from $[\mathrm{Ker}(B)]_\perp$ onto space P_h, where:

$$[\mathrm{Ker}(B)]_\perp = \{v \in V_h, \;\; b(v, p) = 0 \quad \forall p \in P_h\} \subset V_h.$$

Thus, one sees that

$$\dim P_h = \dim[\mathrm{Ker}(B)]_\perp \leq \dim V_h$$

or

$$n \leq m.$$

Using the concept of singular value decomposition from the theory of matrices [11], we may write:

$$B = Q\Sigma S^T \quad (5.4)$$

where Q and S are orthogonal matrices, and Σ is an $(n \times m)$ matrix that has the representation:

$$\Sigma = \begin{pmatrix} \mu_1 & 0 & 0\cdots0 \\ & \mu_2\ 0 & 0\ 0 \\ & \cdot\ \cdot & \cdot\ 0 \\ & \cdot & \cdot\ 0 \\ 0 & 0\cdots\mu_K & \cdot\ 0 \\ 0 & 0\cdots\cdot & 0\cdot 0 \\ 0 & 0\cdots\cdot & 0\cdot 0 \\ 0 & 0\cdots\cdot & 0\ 0 \end{pmatrix}_{(n\times m)} . \tag{5.5}$$

Here μ_i is the ith singular value of the rectangular matrix B and, by definition [11], is determined by:

$$\mu_i = [\lambda_i(B^TB)]^{1/2} \qquad i = 1, \ldots k \tag{5.6}$$

where $\lambda_i(A)$ denotes the ith eigenvalue of the symmetric matrix A, and k is the rank of B. Thus, we have:

$$\frac{b(v, p)}{\| v \|_V \| p \|_P} = \frac{\langle (Q\Sigma S^Tv), p\rangle}{\| v |_V \| p \|_P} = \frac{\langle (\Sigma S^Tv), Q^Tp\rangle}{\| Q^Tp \|_P \| S^Tv \|_V}, \quad \forall (v, p) \in V \times P \tag{5.7}$$

since $\| Q^Tp \| = \| p \|$ and $\| S^Tv \| = \| v \|$ for orthogonal matrices Q and S. Hence, the discrete stability condition (5.1) may be expressed, equivalently, as:

$$\inf_{\forall p\in P_h} \sup_{\forall v\in V_h} \frac{\langle \Sigma v, p\rangle}{\| p \|_P \| v \|_V} \geqslant \beta > 0. \tag{5.8}$$

Theorem 5.1 Condition (5.8) holds if, and only if, the following rank condition holds:

$$\text{rank}\ (B) = n \leqslant m \tag{5.9}$$

Proof. Suppose (5.9) holds. Then the matrix Σ has the representation:

$$\Sigma = (\Sigma_n, 0)$$

where Σ_n is a diagonal matrix with positive diagonal terms μ_i ($i = 1, \ldots, n$). For a given $p \in P_h$, we take:

$$v_p = \begin{pmatrix} p \\ 0 \end{pmatrix}_{m\times 1}.$$

Thus, we have:

$$\inf_{\forall p\in P_h} \sup_{\forall v\in V_h} \frac{\langle \Sigma v, p\rangle}{\| v \|_V \| p \|_P} \geqslant \inf_{\forall p\in P_h} \frac{\langle \Sigma_n p, p\rangle}{\| p \|_P^2} = \min_{1\leqslant i\leqslant n} \mu_i \triangleq \beta > 0. \tag{5.10}$$

Conversely, suppose that (5.8) holds and the rank of B is less than n. Then Σ must be of the form:

$$\Sigma = \begin{pmatrix} \mu_1 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0\cdots\mu_K & & 0 & 0 \\ & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot \\ 0\cdots & \cdots 0 & & 0 \end{pmatrix}_{n\times m} \qquad k < n.$$

We may take $\bar{p} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ P_n \end{pmatrix} n \times 1$ with $p_n \neq 0$.

Then we have: $\langle \Sigma v, p \rangle = \langle v, \Sigma^T \bar{p} \rangle = 0$, since $\Sigma^T \bar{p} = 0$. It can be seen that this leads to a contradiction of eqn (5.8). Then we have rank $(B) \geq n$, or equivalently, rank $(B) = n$.

Remark

One can prove [12] a result stronger than (5.10), as follows:

$$\inf_{\forall p \in P_h} \sup_{\forall v \in V_h} \frac{b(v, p)}{\| v \|_V \| p \|_P} = \min_i (\mu_i). \tag{5.11}$$

Thus, eqn (5.11) suggests that, from the viewpoint of stability, it is preferable to construct a hybrid-mixed finite element model such that the smallest singular value of the matrix B is as large as possible.

6. GLOBAL AND LOCAL STABILITY CONDITIONS OF THE "HYBRID-STRESS" AND "MIXED" FINITE ELEMENT METHODS BASED ON REISSNER'S PRINCIPLE

From Reissner's principle, modified to account for fields σ_{ij} and u_i that do not obey inter-element traction-reciprocity or displacement compatibility as in eqns (2.29), (2.31) and (2.33), one may derive a variety of finite element methods. This variety becomes much more diverse in the case of problems of plates and shells wherein the three-displacement components u_i are subject to certain plausible constraints. For instance, if u_α ($\alpha = 1, 2$) are displacements in the plane of a thin plate, and u_3 is in the normal direction, the well-known hypothesis of Kirchhoff implies that: $u_\alpha = u_{\alpha 0} - x_3(\partial u_{30}/\partial x_\alpha)$. Thus, inter-element continuity of u_i then necessitates the continuity of $u_{\alpha 0}$, u_{30}, and $(\partial u_{30}/\partial x_\alpha)$, where u_{i0} are displacements at the mid-plane of the plate. In a finite element approximation, with arbitrary or even simple-shaped finite elements, it is often inconvenient, if not impossible, to choose a u_{30} such that it and its first derivatives *w.r.t.* x_α are continuous at ρ_m. On the other hand, it is easier to assume $u_{\alpha 0}$ that are continuous at inter-element boundaries. Thus, these situations call for modifications to Reissner's principle only to account for possible discontinuities in the assumed u_{30} and/or $\partial u_{30}/\partial x_\alpha$. These "partial modifications" are discussed in detail in Atluri and Pian [13], and references cited therein.

We shall now consider two specific examples: (i) a mixed finite element method, wherein an arbitrary, symmetric, σ_{ij} [not satisfying eqn (2.8)] and a compatible u_i [satisfying eqn (2.9) and (2.3b)] are assumed in each Ω_m and the finite element method is based on (2.26); and (ii) a hybrid finite element method, wherein an *equilibrated* σ_{ij} [not satisfying eqn (2.8)] is assumed in each Ω_m, and an inter-element-compatible displacement field [satisfying eqns (2.9) and (2.3b)] is assumed at $\partial\Omega_m$ only.

The variational basis of the above mixed method may be seen, from eqn (2.26), to be:

$$0 = \sum_m \left\{ \int_{\Omega_m} \left[\left(u_{(i,j)} - \frac{\partial W_c}{\partial \sigma_{ij}} \right) \tau_{ij} + \sigma_{ij} v_{(i,j)} - \bar{f}_i v_i \right] dv - \int_{S_{tm}} \bar{t}_i v_i \, ds \right\} \tag{6.1}$$

or, in abstract form:

Find $(\sigma, u) \in T \times V_{c0}$ such that:

$$\begin{aligned} a(\sigma, \tau) - b(\tau, u) &= 0 \quad \forall \tau \in T \\ b(\sigma, v) &= \langle f, v \rangle \quad \forall v \in V_{c0} \end{aligned} \tag{6.2}$$

where

$$V_{c0} = \{v \in H'(\Omega_m); \quad v_i^+ = v_i^- \text{ at } \rho_m; \quad v_i = 0 \text{ at } S_{um} \; \forall m\} \tag{6.3}$$

Likewise, the variational basis of the above "hybrid-stress" method may be seen, from either eqn (2.26) or (2.30) [since u_i satisfies eqn (2.9), it automatically implies that $u_i^+ = u_i^- = u_i \equiv \bar{u}_{ip}$ at ρ_m], to be:

$$0 = \sum_m' \left\{ \int_{\Omega_m} (\;)\mathrm{d}v - \int_{S_{tm}} (\;)\mathrm{d}s \right\} \tag{6.4}$$

wherein the integrands over Ω_m and S_{tm} are identical to those in eqn (6.1), except now the nature of functions τ_{ij} are different from those in eqn (6.1). This can be emphasized in the abstract form:

Find $(\sigma, u) \in T_0 \times V_{c0} \quad s \cdot t$

$$\begin{aligned} a(\sigma, \tau) - b(\tau, u) &= 0 \qquad \forall \tau \in T_0 \\ B(\tau, v) &= \langle f, v \rangle \qquad \forall v \in V_{c0} \end{aligned} \tag{6.5}$$

where:

$$T_0 = \{\tau_{ij} \in H'(\Omega_m); \quad \tau_{ij,j} = 0, (\sigma_{ij,j} + \bar{f}_i = 0), \text{ in } \Omega_m \; \forall m\} \tag{6.6}$$

and V_{c0} is defined in eqn (6.3). The appropriate functional $\mathcal{L}$ in both the cases is the same as that given in eqn (2.27), except for the integral over S_u which is now set to zero. Inasmuch as problems eqns (6.2) and (6.5), even though having different physical interpretations, are similar in abstract form, we shall study eqn (6.5) henceforth in some detail. According to Theorem 4.1, problem (6.5) has a unique solution if the following conditions are met:

$$\sup_{\forall \sigma \in T_0} \frac{\sum_m \int_{\Omega_m} \sigma_{ij} v_{(i,j)} \, \mathrm{d}\Omega}{\| \sigma_{ij} \|_T} \geqslant \beta \, \| v \|_V \qquad \forall v \in V_{c0} \tag{6.7}$$

and

$$W_c(\sigma_{ij}, \sigma_{ij}) \geqslant \alpha \, \| \sigma \|_T \qquad \forall \sigma \in \mathrm{Ker}(B) \tag{6.8}$$

where α and β are positive, and Ker(B) is defined as:

$$\mathrm{Ker}(B) = \left\{ \sigma \in T_0, \; \sum_m \int_{\Omega_m} \sigma_{ij} v_{(i,j)} \, \mathrm{d}\Omega = 0, \quad \forall v \in V_{c0} \right\} \tag{6.9a}$$

since $\sigma \in T_0$, eqn (6.9) may be written as:

$$\mathrm{Ker}(B) = \left\{ \sigma \in T_0, \; \sum_m \int_{\partial\Omega_m} \sigma_{ij} n_j v_i = 0, \quad \forall v \in V_{c0} \right\}. \tag{6.9b}$$

Let the dimension of T_0 be m and that of V_{c0} be n. As shown in Section 5, conditions (6.7) and (6.8) imply that $m > n$, and that the rank of the $(n \times m)$ matrix B in

$$v^t B \sigma = \sum_m \int_{\Omega_m} \sigma_{ij} v_{(i,j)} \, \mathrm{d}\Omega \tag{6.10}$$

should be n.

The condition for existence and uniqueness of the solution may also be stated, from eqn (6.7), as

$$\sup_{\forall \sigma \in T_0} \sum_m \int_{\Omega_m} \sigma_{ij} v_{(i,j)} \, d\Omega > 0 \qquad \forall v \in V_{c0} \tag{6.11}$$

since $v_i = 0$ at S_{um} for $v \in V_{c0}$, it can be seen that v_i appearing (6.11) do not include any global rigid body modes, assuming that the original boundary conditions at S_u are such that they preclude any rigid motion of the solid as a whole.

However, even while global rigid motions may be precluded, such rigid motion may be considered at the element level. Thus, $v^m_{(i,j)} = 0$ for r possible rigid modes of each element-displacement field, v^m. Furthermore, σ^m_{ij} is arbitrary for each element, while within each element, $\sigma^m \in T_0$. Thus, eqn (6.11) may be written as:

$$\sup_{\forall \sigma \in T_0} \sum_m \int_{\Omega_m} \sigma^m_{ij} v^{md}_{(i,j)} \, d\Omega > 0 \qquad \forall v \in V_{c0} \tag{6.12}$$

where v^{md}_i denote non-rigid deformation modes in each element. A sufficient condition for the validity of eqn (6.12) is, clearly,

$$\sup_{\forall \sigma \in T_0} \int_{\Omega_m} \sigma^m_{ij} v^{md}_{(i,j)} \, d\Omega > 0 \qquad \forall v \in V_{c0}, \forall_m. \tag{6.13}$$

Let M_β be the number of stress modes assumed in each element Ω_m, and let N_q be the number of displacement modes for each Ω_m. Then the dimension of v^{md}_i is $(N_q - r)$. Thus, we see

$$\begin{aligned} &\int_{\Omega_m} \sigma^m_{ij} v^{md}_{(i,j)} \, d\Omega = \sigma^{mt} B^{*mt} v^{md} \\ &(1 \times M_\beta)(M_\beta \times N_q - r)(N_q - r \times 1). \end{aligned} \tag{6.14}$$

From theorem (5.1) it then follows that, for eqn (6.13) to hold, $M_\beta \geqslant (N_q - r)$ and that the rank of B^{*m} should be $(N_q - r)$.

Remark 6.1

If $a(\sigma, \sigma)$ in each element, $\forall \sigma \in T_0$, [denoted as $a_m(\sigma, \sigma)$] can be written as

$$a_m(\sigma, \sigma) = \underset{(1 \times M^\beta)}{\sigma^{mt}} \; \underset{(M_\beta \times M_\beta)}{H} \; \underset{(M_\beta \times 1)}{\sigma^m}$$

then it can be shown [14] that the element stiffness matrix k_m can be written as:

$$k_m = B^m H^{-1} B^{mt}$$

where B^m is defined through

$$\int_{\Omega_m} \sigma^m_{ij} v^m_{(i,j)} \, d\Omega = \sigma^{mt} B^{mt} v^m.$$

Here, v^m includes both rigid and non-rigid modes, and dim $(v^m) = N_q$. Note that the rank of B^m is the same as that of B^{*m}. Since $a_m(\sigma, \sigma)$ is positive definite, it follows that the rank of the element stiffness matrix is

$(N_q - r)$, provided that of B^m is $(N_q - r)$.

Remark 6.2

Note that both σ_{ij} and v_i in eqn (6.14) are components in the cartesian system x_i. The momentum balance condition involves differentiation of σ_{ij} *w.r.t* x_j; while the strains $v_{(i,j)}$ also involve differentiation *w.r.t* x_j. In the usual *isoparametric* element formulation, the geometrical transformation between the (nondimensional) "parent" element and that in the physical domain is $x_j = x_j(\xi^k)$ where ξ^k, usually taken to be $-1 \leq \xi^k \leq 1$, are curvilinear coordinates. In a displacement formulation, one usually assumes $u_i = u_i(\xi^k)$, and in an isoparametric representation, the representation for x_j as well as u_i contain an equal number of basis functions in ξ^k. The stiffness matrix of the element in the displacement formulation, which depends on $W(\partial v_i/\partial x_k)$, can be shown to be *objective* or observer invariant.

That means, if k_m is the element stiffness matrix in an isoparametric displacement formulation in the x_i coordinate system, then its representation in any other cartesian system $x' = Qx$ is given by $k_m' = Qk_mQ^T$, where Q is orthogonal.

To maintain the objectivity of the element stiffness matrix in a mixed-hybrid formulation, it has been shown [15–17] that the stress tensor, $\boldsymbol{\sigma}$, should be assumed in an element-local coordinate system and *not* in a global coordinate system.

Remark 6.3

First consider two- and three-dimensional finite elements of square and cubic (or rectangular and rectangular prism) shapes, respectively. Here, the theory of symmetric groups has been demonstrated [15–17] to be a useful tool in choosing least-order stress-fields ($M_\beta = N_q - r$) that lead to the matrix $B^{*'m}$ (see eqn 6.14) of rank ($N_q - r$), and a stiffness matrix k_m, which is objective and also of rank ($N_q - r$). In this case, a cartesian coordinate system located at the centroid of the element, and along the axes of symmetry of the element, are used. In [15, 16], both σ_{ij} and $v_{(i,j)}$ [$(\ )_{,j}$ implying $\partial(\ )/\partial x_j$] are decomposed into invariant irreducible spaces using group theory. In terms of these irreducible representations, the matrix, say $(B^*)'$, corresponding to $B(\sigma, v)$ for each element, becomes "quasi-diagonal." Thus, group theory enables one to pick σ_{ij} in each element, for a given v_i, such that the resulting element formulation is invariant and stable. It has been shown [16, 17] that: (i) for a four-noded square with $N_q - r = 5$, there are two possible choices for a five-parameter equilibrated stress field; (ii) for an eight-noded square with $N_q - r = 13$, there are 21 choices for a 13-parameter stress field; (iii) for an eight-noded cube, there are eight choices for a stress field with $M_\beta = 18$; and (iv) for a 20-noded cube, there are 384 choices for a stress field with $M_\beta = 54$; all of which lead to stable and objective elements. The 'best' selection among all these choices may depend upon: (i) the lowest eigenvalue of the matrix $(B^{*'m})(B^{*m'})$ as discussed in Section 5 and (ii) the capability of the candidate stress field to represent the cardinal states of stress of pure tension, shear, bending, and torsion in each element. A comprehensive study of such tests is given in [16, 17].

Remark 6.4

Consider a mixed-hybrid element of a general curvilinear shape and introduce a geometric mapping of the type $x_i = x_i(\xi^k)$, with $-1 \leq \xi^k \leq 1$. Let $\mathbf{g}_k(\xi^m)$ and $\mathbf{g}_k(\xi^k)$ be the covariant and contravariant base vectors, respectively, of the curvilinear coordinates ξ^m. Let $\bar{\mathbf{g}}_k$ represent the covariant base-vectors at the centroid, i.e. $\bar{\mathbf{g}}_k = \mathbf{g}_k(\xi^m = 0)$, and let $\mathbf{e}_k$ be a cartesian system at $\xi^m = 0$. Then, it has been shown [15, 16] that requirements of invariance may be met by representing the stress tensor in the alternative forms:

$$\boldsymbol{\sigma} = \sigma_{ij}(x_k)\mathbf{e}_i\mathbf{e}_j, \quad \sigma_{ij} \in T_0 \tag{6.15a}$$

$$= \sigma_{ij}(\xi^k)\mathbf{e}_i\mathbf{e}_j, \quad \sigma_{ij} \in T \tag{6.15b}$$

$$= \sigma_{ij}(\xi^k)\mathbf{g}^i\mathbf{g}^j, \quad \sigma_{ij} \in T \tag{6.15c}$$

$$= \sigma^{ij}(\xi^k)\mathbf{g}_i\mathbf{g}_j, \quad \sigma^{ij} \in T \tag{6.15d}$$

$$= \sigma^{ij}(\xi^k)\bar{\mathbf{g}}_i\bar{\mathbf{g}}_j, \quad \sigma^{ij} \in T. \tag{6.15e}$$

Other possible representations are discussed in [18]. In eqn (6.15) T_0 is the space of *equilibrated* stresses, while T is that of differentable (but not equilibrated) stresses. It is seen that (6.15a and b) can easily represent states of constant stress in the cartesian coordinate system and hence can pass the so-called "constant-stress" patch test [19].

Considering a state of constant stress, say, $\boldsymbol{\sigma} = C_{ij}\mathbf{e}_i\mathbf{e}_j$, where C_{ij} are constants, it is seen that representation (6.15c) can pass the patch test if $\sigma_{ij}(\xi^k)$ includes functions such that:

$$\sigma_{ij}(\xi) = C_{mn}(\mathbf{e}_m \cdot \mathbf{g}_i)(\mathbf{e}_n \cdot \mathbf{g}_j) \tag{6.16a}$$

$$= C_{mn}\frac{\partial x_m}{\partial \xi^i}\frac{\partial x_n}{\partial \xi^j}. \tag{6.16b}$$

Since, in an isoparametric formulation, $(\partial x_m/\partial \xi^i)$ is a simple polynomial in ξ^k, it is possible, in general, that a polynomial representation exists for $\sigma_{ij}(\xi^k)$ in (6.15c) which passes the patch test. However, the stress field will not be, in general, of the 'least-order.'

On the other hand, eqn (6.15d) can pass the patch test if $\sigma^{ij}(\xi^k)$ includes functions such that:

$$\sigma^{ij}(\xi^k) = C_{mn}(\mathbf{e}_m \cdot \mathbf{g}^i)(\mathbf{e}_n \cdot \mathbf{g}^j) \tag{6.17a}$$

$$= C_{mn}\frac{\partial \xi^i}{\partial x_m}\frac{\partial \xi^j}{\partial x_n}. \tag{6.17b}$$

For the usual isoparametric formulation, it is seen that $\sigma^{ij}(\xi^k)$ of eqn (6.17b) are no longer simple polynomials. Hence, representation (6.15d) with polynomial functions $\sigma^{ij}(\xi^k)$ will not, in general, pass the patch test. However, eqn (6.16e) will pass the patch test, since, in this case,

$$\sigma^{ij}(\xi^k) = C_{mn}(\mathbf{e}_m \cdot \bar{\mathbf{g}}_i)(\mathbf{e}_n \cdot \bar{\mathbf{g}}_j) \equiv d_{mn} \tag{6.18}$$

where d_{mn} are constants, and hence a simple (even least-order) polynomial representation (including constant terms) will suffice for $\sigma^{ij}(\xi^k)$.

Remark 6.5

To formulate an isoparametric curvilinear mixed-hybrid element, one may use alternative representations for stress as in eqn (6.15a–e) and assume $v_i(\xi^k)$ to be of the same form as $x_i(\xi^k)$. Note that v_i are cartesian components of displacement. For the alternative representations of stress as in eqn (6.15), the bilinear form $B(\sigma, v)$, for each element, takes on the respective representation:

$$B(\sigma, v) = \int_{\Omega_m} \sigma_{ij}(x_k)v_{i;m}J_{mj}^{-1}(\det J)\, d\xi^1\, d\xi^2\, d\xi^3 \tag{6.19a}$$

$$= \int_{\Omega_m} \sigma_{ij}(\xi^k)v_{i;m}J_{mj}^{-1}(\det J)\, d\xi^1\, d\xi^2\, d\xi^3 \tag{6.19b}$$

$$= \int_{\Omega_m} \sigma_{mn}(\xi^k)v_{i;k}J_{kj}^{-1}J_{mi}^{-1}J_{nj}^{-1}(\det J)\, d\xi^1\, d\xi^2\, d\xi^3 \tag{6.19c}$$

$$= \int_{\Omega_m} \sigma^{mn}(\xi^k)v_{i;n}J_{im}(\det J)\, d\xi^1\, d\xi^2\, d\xi^3 \tag{6.19d}$$

$$= \int_{\Omega_m} \sigma^{mn}(\xi^k)v_{i;k}J_{kj}^{-1}(\bar{J}_{im}\bar{J}_{jn})\det(J)\, d\xi^1\, d\xi^2\, d\xi^3 \tag{6.19e}$$

where $(\ )_{;m}$ denotes $\partial(\ )/\partial\xi^m$; $J_{mj} = (\partial x_m/\partial \xi^j)$ and $\bar{J}_{mj} = J_{mj}(\xi^k = 0)$.

In Remark (6.3) concerning squares and cubes, a group-theoretical method which enables a choice of $\sigma_{ij}(x_k)$, for a given $v_i(x_k)$, that gives the rank $(N_q - r)$ to B^{*m} was described. For such squares and cubes, the bilinear form is computed using eqn (6.14). Comparing eqns (6.14) and (6.19), it can be seen that there exists no simple way of choosing the stress as in eqn (6.15) for curvilinear elements such that the rank of B^{*m} is determined *a priori*. However, it has been demonstrated in [16, 17] that if $\sigma_{ij}(\xi^k)$ or $\sigma^{ij}(\xi^k)$ of eqn (7.15b–e) is chosen to be of the same polynomial form (i.e. by replacing x_k by ξ^k) as that of $\sigma_{ij}(x_k)$ which is derived by using group theory for squares and cubes, then the rank of B^{*m} is maintained to be $(N_q - r)$ even for very severely distorted elements. Further, it has been clearly demonstrated [16, 17] that the least-order, invariant, isoparametric, curvilinear mixed-hybrid elements are less distortion-sensitive and lead to more accurate results compared to the standard displacement elements in a variety of examples.

7. EXAMPLES

Consider a four-noded square element, such that $-1 \leq (x_1, x_2) \leq +1$. The displacement-field is of the form:

$$u_i = a_{0i} + a_{1i}x_1 + a_{2i}x_2 + a_{3i}x_1x_2. \tag{7.1}$$

As shown in [15] from group-theoretical considerations, an equilibrated, least-order (five parameter), invariant stress field that leads to the correct rank (i.e. 5) of the element matrix B^*, is either:

$$\boldsymbol{\sigma} = \left[\begin{array}{c|c} (a_1 + a_2x_1) & (a_7 - a_6x_1 - a_2x_2) \\ & (a_4 + a_6x_2) \end{array}\right] \tag{7.2}$$

or

$$\boldsymbol{\sigma} = \begin{bmatrix} (a_1 + a_3x_2) & a_2 \\ 0 & (a_4 + a_5x_1) \end{bmatrix}. \tag{7.3}$$

For both the cases, the lowest eigenvalue of $(B^{*T}B^*)$ can be shown to be (1/3). Note, however, eqn (7.3) can represent the "bending" stress field, while eqn (7.2) cannot. On the other hand, an equilibrated, seven parameter, invariant stress field that still maintains the rank of B^* to be five can be shown [16] to be:

$$\sigma_{11} = (a_1 + a_2x_1 + a_3x_2);\quad \sigma_{22} = a_4 + a_5x_1 + a_6x_2; \sigma_{12} = a_7 - a_6x_1 - a_2x_2. \tag{7.4}$$

Here, the lowest eigenvalue of $B^{*T}B$ can be shown to be $(\sqrt{2}/3)$. It is also seen that the stress-field eqn (7.4) is complete, in addition to being invariant. Thus, the use of eqn (7.4) may be justified from stability (lowest eigenvalues of $B^{*T}B$ is the largest in magnitude) as well as completeness point of view, even though it may be more expensive in a computational sense. (Note that 'completeness' is not synonymous with invariance.)

Consider now an eight-noded cubic element. Here the displacement field is of the type:

$$u_i = a_{0i} + a_{1i}x_1 + a_{2i}x_2 + a_{3i}x_3 + a_{4i}x_1x_2 + a_{5i}x_1x_3 + a_{6i}x_2x_3 + a_{7i}x_1x_2x_3. \tag{7.5}$$

Here, an 18 ($N_q - r \equiv$ 24-6) parameter stress-field that leads to the rank of 18 for B^{*m}

can be shown [15, 16] to have eight choices, in the form:

$$\boldsymbol{\sigma} = \delta_1 \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} + \delta_2 \begin{bmatrix} 0 & x_3 & x_2 \\ & 0 & x_1 \\ & & 0 \end{bmatrix} + \delta_3 \begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix} + \delta_4 \begin{bmatrix} 0 & & \\ & 1 & \\ & & -1 \end{bmatrix}$$

$$+ \delta_5 \begin{bmatrix} 0 & x_3 & 0 \\ & 0 & -x_1 \\ & & 0 \end{bmatrix} + \delta_6 \begin{bmatrix} 0 & 0 & x_2 \\ & 0 & -x_1 \\ & & 0 \end{bmatrix} + \delta_7 \begin{bmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{bmatrix} + \delta_8 \begin{bmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{bmatrix}$$

$$+ \delta_9 \begin{bmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{bmatrix} \left\{ + \delta_{10} \begin{bmatrix} x_2 & & \\ & 0 & \\ & & x_2 \end{bmatrix} + \delta_{11} \begin{bmatrix} 0 & & \\ & x_1 & \\ & & x_1 \end{bmatrix} + \delta_{12} \begin{bmatrix} x_3 & & \\ & x_3 & \\ & & 0 \end{bmatrix} \right.$$

or

$$\left. + \delta_{10} \begin{bmatrix} 2x_1 & -x_2 & -x_3 \\ & 0 & 0 \\ & & 0 \end{bmatrix} + \delta_{11} \begin{bmatrix} 0 & -x_1 & 0 \\ & 2x_2 & -x_3 \\ & & 0 \end{bmatrix} + \delta_{12} \begin{bmatrix} 0 & 0 & -x_1 \\ & 0 & -x_2 \\ & & 2x_3 \end{bmatrix} \right\}$$

$$+ \left\{ \delta_{13} \begin{bmatrix} x_2 & & \\ & 0 & \\ & & -x_2 \end{bmatrix} + \delta_{14} \begin{bmatrix} 0 & & \\ & x_1 & \\ & & -x_1 \end{bmatrix} + \delta_{15} \begin{bmatrix} x_3 & & \\ & -x_3 & \\ & & 0 \end{bmatrix} \right.$$

or

$$\left. + \delta_{13} \begin{bmatrix} 0 & x_2 & -x_3 \\ & 0 & 0 \\ & & 0 \end{bmatrix} + \delta_{14} \begin{bmatrix} 0 & x_1 & 0 \\ & 0 & -x_3 \\ & & 0 \end{bmatrix} + \delta_{15} \begin{bmatrix} 0 & 0 & x_1 \\ & 0 & -x_2 \\ & & 0 \end{bmatrix} \right\}$$

$$+ \left\{ \delta_{16} \begin{bmatrix} 0 & 2x_1x_3 & 2x_1x_2 \\ & 0 & -(x_2^2 + x_3^2) \\ & & 0 \end{bmatrix} + \delta_{17} \begin{bmatrix} 0 & 2x_2x_3 & -(x_1^2 + x_3^2) \\ & 0 & 2x_1x_2 \\ & & 0 \end{bmatrix} \right.$$

$$+ \delta_{18} \begin{bmatrix} 0 & -(x_1^2 + x_2^2) & 2x_2x_3 \\ & 0 & 2x_1x_3 \\ & & 0 \end{bmatrix}$$

$$\text{or} + \delta_{16} \begin{bmatrix} 0 & & \\ & 0 & \\ & & x_1x_2 \end{bmatrix} + \delta_{17} \begin{bmatrix} 0 & & \\ & x_1x_3 & \\ & & 0 \end{bmatrix} + \left. \delta_{18} \begin{bmatrix} x_2x_3 & & \\ & 0 & \\ & & 0 \end{bmatrix} \right\}. \qquad (7.6)$$

By numbering each choice in each { } as 1 or 2, the eight possible stress fields are obtained by taking {1,1,1}, {1,2,2}, {1,1,2}, {1,2,1}, {2,1,1}, {2,2,2}, {2,1,2}, {2,2,1} in addition to the first nine terms of eqn (7.6). As noted earlier, which one of these eight choices is better than the others depends on: (i) the ability of the chosen field to represent certain cardinal states of stress such as tension, bending, torsion, etc. and (ii) the lowest eigenvalue of $B^{*T}B^*$. Suppose one requires the element to represent "bending," i.e. $\sigma_{11} = a + bx_2 + cx_3 +$ etc. The two choices that include this bending exactly, as seen from eqn (7.5), are: {1,1,1} or {1,1,2}. It can easily be computed that the minimum eigenvalue of the element matrix $B^{*T}B^*$ for the choice {1,1,1} is (1.9/9), while that for {1,1,2} is (1/9). Thus, from the point of view of the aforementioned criteria (i) and (ii), choice {1,1,1} may be seen to be better. Choice {1,1,2} has also been derived independently, from a heuristic reasoning, in [20].

Acknowledgements—The results presented herein were obtained during the course of investigations supported by NASA-Lewis Research Center under grant NAG3-346 to Georgia Tech. The encouragement of Drs. Chris Chamis and Laszlo Berke is thankfully acknowledged. It is a pleasure to record here our thanks to Ms. J. Webb for her expert assistance in the preparation of the typescript.

REFERENCES

1. E. Reissner, On a variational theorem in elasticity. *J. Math. Phys.* **29**, pp. 90–95 (1950).
2. F. Brezzi, On the existence, uniqueness, and approximation of saddle-point problems arising from Lagrange multipliers. R.A.I.R.O. 8-R2, pp. 129–151 (1974).
3. E. Reissner, Variational principles in elasticity in *Handbook of Finite Elements: Part II—Finite Element Methods*. McGraw-Hill (to appear).
4. S. N. Atluri, On hybrid finite element models in solid mechanics in *Advances in Computer Methods for Partial Differential Equations* (Edited by R. Vichnevetsky), *AICA*, Rutgers University (USA)/University of Ghent (Belgium), pp. 346–356 (1975).
5. S. N. Atluri, P. Tong, and H. Murakawa, Recent studies in hybrid and mixed finite element methods in mechanics in *Hybrid and Mixed Finite Element Methods* (Edited by S. N. Atluri, R. H. Gallagher and O. C. Zienkiewicz), pp. 51–71. John Wiley (1983).
6. I. Babuska, J. T. Oden, and J. K. Lee, Mixed-hybrid finite element approximations of second-order elliptic boundary-value problems Part I in *Comp. Meths. Appl. Mech. Engrg* **11**, pp. 175–206 (1977); Part II—Weak-hybrid Methods, **14**, pp. 1–22 (1978).
7. O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*. Gordon and Breach, New York (1969).
8. I. Babuska and A. K. Aziz, Survey lectures on the mathematical foundations of the finite element method in *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations* (Edited by A. K. Aziz), Academic Press, New York (1972).
9. V. Girault and P. A. Raviart, *Finite Element Approximations of the Navier-Stokes Equations*. Lecture Notes in Mathematics, Vol. 479, Springer-Verlag (1979).
10. L-A. Ying and S. N. Atluri, A Hybrid Finite Element Method for Stokes Flow: Part II—Stability and Convergence Studies. *Comp. Meth. Appl. Mech. Engng* **36**, pp. 39–60 (1983).
11. H. Schwerdfeger, *Introduction to Linear Algebra and the Theory of Matrices*. Noorhoff, Leyden, 2nd Ed. (1961).
12. W-M. Xue, Existence, uniqueness, and stability conditions for general mixed-hybrid finite element methods in linear elasticity, Ph.D. thesis, Georgia Institute of Technology (1984) (in preparation).
13. S. N. Atluri and T. H. H. Pian, Theoretical formulation of finite element methods in linear elastic analysis of general shells. *J. Struct. Mech.* **1**, pp. 1–41 (1972).
14. T. H. H. Pian and P. Tong, Basis of finite element methods for solid continua. *Int. J. Num. Meth. Engng* **1**, pp. 3–28 (1969).
15. R. Rubenstein, E. F. Punch, and S. N. Atluri, An Analysis of, and Remedies for, Kinematic Modes in Hybrid-Stress Finite Elements: Selection of Stable, Invariant Stress Fields. *Comp. Meth. Appl. Mech. Engng* **38**, pp. 63–92 (1982).
16. E. F. Punch, Stable, invariant, least-order isoparametric mixed-hybrid stress elements: linear elastic continua, and finitely deformed plates and shells. Ph.D. thesis, Georgia Institute of Technology (Aug. 1983).
17. E. F. Punch and S. N. Atluri, Development and testing of stable, invariant, isoparametric curvilinear 2- and 3-D hybrid stress elements. *Comp. Meth. Appl. Mech. Engng* (1984) (submitted).
18. S. N. Atluri, H. Murakawa, and C. Bratianu, Use of stress functions and asymptotic solutions in finite element analysis of continua in *New Concepts in Finite Element Analysis ASME-AMD* Vol. 44, pp. 11–29 (1981).
19. B. M. Irons and S. Ahmad, *Techniques of Finite Element Analysis*, Ellis Horwood Limited (1980).
20. T. H. H. Pian and D. P. Chen, On the Suppression of Zero Energy Deformation Modes, *Int. J. Num. Meth. Engng* **19**, 12, pp. 1741–1752 (1983).